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NOTE

Conservative and Orthogonal Discretization for the Lorentz Force

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1. INTRODUCTION

Brackbill and Barnes [2] argued that in the presence of errors in $\nabla \cdot \mathbf{B}$, the Lorentz force $\mathbf{J} \times \mathbf{B}$ should be discretized in such a way that it is orthogonal to the magnetic field $\mathbf{B}$ in a discrete sense. Here $\mathbf{B}$ is the magnetic field and $\mathbf{J} = \nabla \times \mathbf{B}$ is the current density. This is relatively easy to achieve if $\mathbf{J} \times \mathbf{B}$ is evaluated as a source term. Many magnetohydrodynamics (MHD) codes, for example the ZEUS code [8], follow this approach. Unfortunately such discretizations are not conservative in general. The conservative discretizations, on the other hand, use the divergence of the Maxwell stress tensor, but then the orthogonality property is not easy to satisfy even if $\nabla \cdot \mathbf{B} = 0$ in some discrete sense.

In a recent paper [10] I tried to prove that the requirement that the discretization of the Lorentz force be orthogonal to the magnetic field cannot be met in conservative schemes. The counterexample shown in that paper, however, does not prove this. In the counterexample I set up a magnetic field which has zero divergence and the x component of the Lorentz force is zero everywhere except in one cell. This corresponds to the magnetic field distribution shown in Fig. 1. Unfortunately the conclusion that this would contradict the conservation of the x component of the momentum is incorrect. In fact, a conservative scheme can accelerate the flow in the middle of the computational domain due to the inbalance of forces at the boundaries, which in this case comes from the Maxwell stress tensor.

Further reflection on the problem revealed that it is in fact possible to construct a conservative scheme in which the discretized Lorentz force is orthogonal to the magnetic field in the cell centers. In this note I will show the requirements for such a discretization and a possible realization. This note does not try to evaluate the accuracy of such discretizations; I am simply showing the existence of, and conditions for, such schemes.

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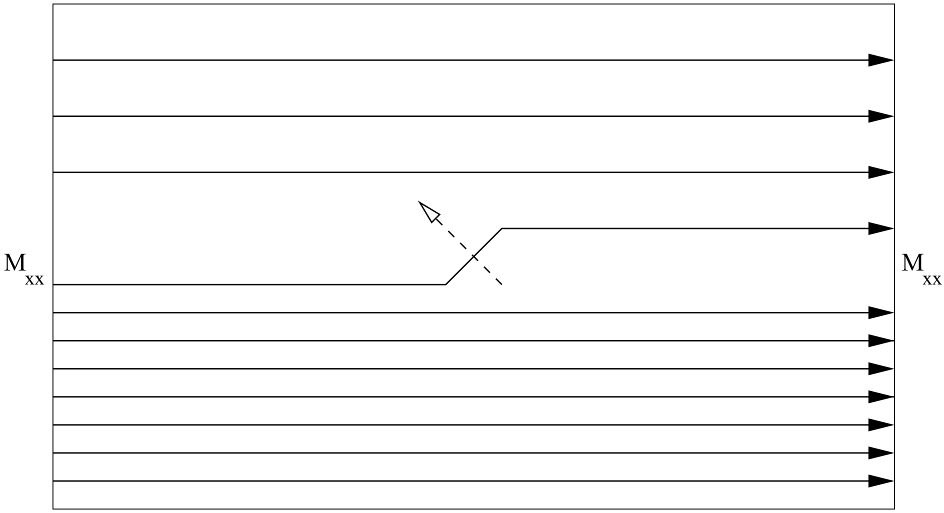

FIG. 1. Schematic plot of the magnetic field (arrows) for the counterexample presented in [10]. The Lorentz force (dashed arrow) accelerates the flow to the left, but this does not contradict momentum conservation, because the $M_{x,x}$ component of the Maxwell stress tensor differs at the left and right boundaries.

2. ANALYTIC CONDITIONS

A conservative finite volume discretization can be written as the sum of numerical fluxes acting on a cell. Therefore the conservative discretization should correspond to the divergence form of momentum equation

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) + \nabla p + \nabla \cdot \left(I \frac{\mathbf{B}^2}{2} - \mathbf{B}\mathbf{B} \right) = 0, \tag{1}$$

where $\mathbf{u}$, $\rho\mathbf{u}$, p, and I are the velocity, momentum density, pressure, and identity matrix, respectively. For the sake of simplicity the magnetic units are chosen such that the vacuum permeability $\mu_0 = 1$. The last term in (1) is the divergence of the Maxwell stress tensor, which is identical to the Lorentz force $\mathbf{J} \times \mathbf{B}$ as long as $\nabla \cdot \mathbf{B} = 0$. The orthogonality condition can be written as

$$0 = \mathbf{B} \cdot \nabla \cdot \left(I \frac{\mathbf{B}^2}{2} - \mathbf{B}\mathbf{B} \right). \tag{2}$$

The discrete equivalent of this equation requires that $\mathbf{B}$ be orthogonal to the discrete form of the divergence of the Maxwell stress tensor in the form it is applied in the momentum density update.

To motivate the discretization of the orthogonality condition let us write (2) in 2D Cartesian coordinates:

$$0 = B_x \left[\frac{1}{2} \partial_x \left(B_y^2 - B_x^2 \right) - \partial_y (B_x B_y) \right] + B_y \left[\frac{1}{2} \partial_y \left(B_x^2 - B_y^2 \right) - \partial_x (B_x B_y) \right]. \tag{3}$$

With fairly simple manipulations (3) becomes

$$0 = -\left(B_x^2 + B_y^2\right)\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y}\right), \tag{4}$$

which holds as long as $\nabla \cdot \mathbf{B} = 0$.

3. DISCRETIZED CONDITIONS

For the sake of simplicity a 2D Cartesian grid is examined. The generalization to 3D is straightforward. The finite volume discretization of the orthogonality condition (2) is

$$\begin{aligned} 0 = B_{i,j}^x&\left(\frac{F_{i+1/2,j}^x - F_{i-1/2,j}^x}{\Delta x} + \frac{F_{i,j+1/2}^y - F_{i,j-1/2}^y}{\Delta y}\right) \\ + B_{i,j}^y&\left(\frac{G_{i+1/2,j}^x - G_{i-1/2,j}^x}{\Delta x} + \frac{G_{i,j+1/2}^y - G_{i,j-1/2}^y}{\Delta y}\right), \end{aligned} \tag{5}$$

where F and G are the magnetic contributions to the fluxes corresponding to ρu_x and ρu_y, respectively.

The discrete fluxes can always be written in the same form as the corresponding analytic expressions, since $\mathbf{B}$ has two independent components at all four interfaces, and there are also two flux components (one for ρu_x and one for ρu_y) at every interface; so

$$F_{i+1/2,j}^x = \frac{1}{2}\left[-\left(B_{i+1/2,j}^x\right)^2 + \left(B_{i+1/2,j}^y\right)^2\right], \tag{6a}$$

$$G_{i+1/2,j}^x = -B_{i+1/2,j}^x B_{i+1/2,j}^y, \tag{6b}$$

$$F_{i,j+1/2}^y = -B_{i,j+1/2}^x B_{i,j+1/2}^y, \tag{6c}$$

$$G_{i,j+1/2}^y = \frac{1}{2}\left[\left(B_{i,j+1/2}^x\right)^2 - \left(B_{i,j+1/2}^y\right)^2\right]. \tag{6d}$$

Before I proceed and substitute the above formulas into the orthogonality condition (5), let me introduce some notation for directional averages and differences; i.e.,

$$\mu_x f = \frac{1}{2}\left(f_{i+1/2,j} + f_{i-1/2,j}\right), \tag{7a}$$

$$\mu_y f = \frac{1}{2}\left(f_{i,j+1/2} + f_{i,j-1/2}\right), \tag{7b}$$

$$\delta_x f = f_{i+1/2,j} - f_{i-1/2,j}, \tag{7c}$$

$$\delta_y f = f_{i,j+1/2} + f_{i,j-1/2}, \tag{7d}$$

where f is some arbitrary variable. Simple algebra shows that the relationships

$$f_{i+1/2,j}^2 - f_{i-1/2,j}^2 = 2\mu_x f \delta_x f, \tag{8a}$$

$$f_{i,j+1/2}^2 - f_{i,j-1/2}^2 = 2\mu_y f \delta_y f, \tag{8b}$$

$$(fg)_{i+1/2,j} - (fg)_{i-1/2,j} = \mu_x f \delta_x g + \delta_x f \mu_x g, \tag{8c}$$

$$(fg)_{i,j+1/2} - (fg)_{i,j-1/2} = \mu_y f \delta_y g + \delta_y f \mu_y g \tag{8d}$$

hold for arbitrary f and g variables. Using the above expressions, the finite volume discretization of the orthogonality condition (5) becomes

$$0 = B^x\left[\frac{\delta_x B^y \mu_x B^y - \delta_x B^x \mu_x B^x}{\Delta x} - \frac{\mu_y B^x \delta_y B^y + \delta_y B^x \mu_y B^y}{\Delta y}\right]$$
$$+ B^y\left[\frac{\mu_y B^x \delta_y B^x - \mu_y B^y \delta_y B^y}{\Delta y} - \frac{\mu_x B^x \delta_x B^y + \delta_x B^x \mu_x B^y}{\Delta x}\right], \tag{9}$$

where B^x and B^y (without any δ or μ in front of them) are short notations for the cell-centered components $B^x_{i,j}$ and $B^y_{i,j}$, respectively. Equation (9) is a general finite volume discretization of the analytic orthogonality condition (3) for a 2D Cartesian grid. The discrete condition can be manipulated into terms approximating $-\mathbf{B}^2\nabla\cdot\mathbf{B}$, as in (4), but several remainder terms also appear:

$$0 = -(B^x\mu_x B^x + B^y\mu_y B^y)\left[\frac{\delta_x B^x}{\Delta x} + \frac{\delta_y B^y}{\Delta y}\right] \tag{10a}$$
$$+ B^y(\mu_y B^y - \mu_x B^y)\frac{\delta_x B^x}{\Delta x} + B^x(\mu_x B^x - \mu_y B^x)\frac{\delta_y B^y}{\Delta y} \tag{10b}$$
$$+ (B^x\mu_x B^y - B^y\mu_x B^x)\frac{\delta_x B^y}{\Delta x} + (B^y\mu_y B^x - B^x\mu_y B^y)\frac{\delta_y B_x}{\Delta y}. \tag{10c}$$

3.1. Sufficient Condition

Looking at the terms in (10), it is easy to find sufficient conditions for the discrete orthogonality to hold:

$$0 = \frac{\delta_x B^x}{\Delta x} + \frac{\delta_y B^y}{\Delta y}, \tag{11a}$$
$$B^x = \mu_x B^x = \mu_y B^x, \tag{11b}$$
$$B^y = \mu_x B^y = \mu_y B^y. \tag{11c}$$

The first condition simply requires that $\nabla\cdot\mathbf{B}$ be zero in this particular discretized form, while the other two conditions mean that the x- and y-averaged face-centered $\mathbf{B}$ fields should give the same cell-centered value.

3.2. Necessary Condition

It is probably not possible to make the conditions necessary for orthogonality much more precise than (10). With some plausible assumptions, however, it can be shown that any reasonably simple discretization must satisfy conditions (11).

Even analytically, the orthogonality condition (3) requires that $\nabla\cdot\mathbf{B} = 0$, so $\nabla\cdot\mathbf{B} = 0$ must hold in some discrete sense too. It seems very reasonable to choose (11a) as the discrete form, since it cancels out the terms in (10a). Next we can take a case when all the field lines are parallel, so analytically $B_y = \alpha B_x$, where α is some constant. Any reasonable discretization should have the same proportionality in the discrete form too, so $B^y = \alpha B^x$ at the cell centers as well as at the cell interfaces. Since both the difference δ and average μ

operators are linear, $B^x \mu_x B^y = \alpha B^x \mu_x B^x = B^y \mu_x B^x$ and similarly $B^x \mu_y B^y = B^y \mu_y B^x$; so the terms in (10c) also cancel out. The remaining terms in (10b) simplify to

$$\begin{aligned} 0 &= \left(\alpha^2 \frac{\delta_x B^x}{\Delta x} - \frac{\delta_y B^y}{\Delta y} \right) B^x (\mu_x B^x - \mu_y B^x) \\ &= (\alpha^2 + 1) \frac{\delta_x B^x}{\Delta x} B^x (\mu_x B^x - \mu_y B^x). \end{aligned} \tag{12}$$

In the second equality we made use of the discrete divergence-free condition (11a). From (12) only the last term $(\mu_x B^x - \mu_y B^x)$ can be zero in general, so (11b) must hold. For the case considered, condition (11c) can be obtained by multiplying (11b) by α.

We have shown that at least for the parallel field line case, assuming that $\nabla \cdot \mathbf{B} = 0$ is discretized as (11a), (11b), and (11c) must be satisfied too. It is now plausible to assume that the same conditions will also hold for more general magnetic field configurations if the discretization is not overly complicated.

4. DISCRETIZATION OF B

Constructing a scheme that satisfies conditions (11) does not seem to be easy at first. The main difficulty is in making the x and y directional averages of the face-centered values the same. The first ad hoc idea could be to define the face-centered values as some linear reconstructions based on the cell-centered values. This would ensure that the average of the face-centered values agrees with the cell-centered value in both the x and y directions. However, the face-centered values would be obtained on a cell by cell basis, so neighboring cells would produce multiple values for the interface between them. Unfortunately, in a conservative scheme the momentum flux has to be uniquely defined at the interface, so the linear reconstruction idea does not work out.

A more systematic approach is to define the face-centered values on appropriate stencils. The stencil must be symmetric with respect to the interface (the weights may not be symmetric), and the combined stencil for $\mathbf{B}_{i-1/2,j} + \mathbf{B}_{i+1/2,j}$ must be the same as the combined stencil for $\mathbf{B}_{i,j-1/2} + \mathbf{B}_{i,j+1/2}$. The simplest solution is shown in Fig. 2. The stencil for each interface-centered value consists of the two endpoints of the interface. This means that the magnetic field is discretized at the corners of the computational cells, while the momentum is discretized in the cell centers. In 3D the situation is the same: the simplest stencil for the face-centered variables consists of the four corners belonging to the face. We have considered this type of vertex-based discretization [11] in the context of prolongation and restriction operators. Note that this staggering is different from the standard constrained transport (CT) discretizations [1, 3–6], where the primary magnetic field variables are located at the face centers, but it strongly resembles (except for a diagonal shift by $\Delta x/2$, $\Delta y/2$) the finite volume interpretation of CT schemes [10], where the magnetic field is discretized in the cell centers.

The simplest choice for the interface-centered variables is to define them as the average of the corners belonging to the interface; i.e.,

$$\mathbf{B}_{i+1/2,j} = \frac{\mathbf{B}_{i+1/2,j+1/2} + \mathbf{B}_{i+1/2,j-1/2}}{2}, \tag{13a}$$

$$\mathbf{B}_{i,j+1/2} = \frac{\mathbf{B}_{i+1/2,j+1/2} + \mathbf{B}_{i-1/2,j+1/2}}{2}. \tag{13b}$$

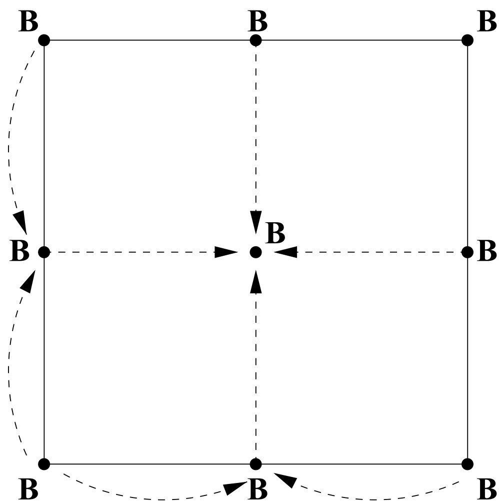

FIG. 2. The simplest discretization of the magnetic field that satisfies the conditions (11b) and (11c). The face-centered **B** is an average of the corner values; thus the same cell-centered **B** is obtained independently regardless of whether the x or the y face values are averaged.

This definition automatically satisfies conditions (11b) and (11c), since the cell-centered value will be the average of the values at the corners of the cell independent of the direction in which the face values are averaged. The only problem remaining is to satisfy the divergence-free condition (11a).

5. CONSTRAINED TRANSPORT FOR VERTEX-BASED B

There are various ways to enforce (11a) for the magnetic field, which is updated by the induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}, \tag{14}$$

where $\mathbf{E} = -\mathbf{u} \times \mathbf{B}$ is the electric field in the MHD approximation. The projection scheme is a general, although somewhat expensive, solution. In the context of the current paper, a constrained transport-type solution fits the rest of the algorithm much better. To construct such a scheme, again we need to consider the stencils belonging to the update of the magnetic field in the cell corners.

Divergence **B** in the sense of (11a) can be conserved by the induction equation (14) if the stencil for $\nabla \times \mathbf{E}$ is symmetric around the cell corners. The simplest choice is shown in Fig. 3. The discretized induction equation will be

$$B^{x,n+1}_{i+1/2,j+1/2} = B^{x,n}_{i+1/2,j+1/2} - \Delta t \frac{E^z_{i,j+1} + E^z_{i+1,j+1} - E^z_{i,j} - E^z_{i+1,j}}{2\Delta y}, \tag{15a}$$

$$B^{y,n+1}_{i+1/2,j+1/2} = B^{y,n}_{i+1/2,j+1/2} + \Delta t \frac{E^z_{i,j+1} + E^z_{i+1,j+1} - E^z_{i,j} - E^z_{i+1,j}}{2\Delta x}. \tag{15b}$$

It is easy to show that this update conserves the divergence as defined in (11a) if the face values are defined by (13). This algorithm is analogous to the finite volume interpretation of CT schemes described in [10].

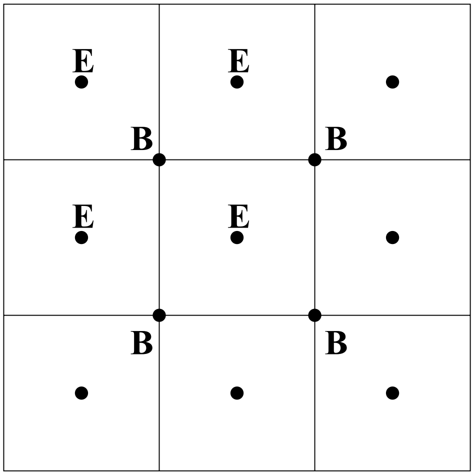

FIG. 3. The simplest constrained transport-type discretization of the induction equation for the cell-corner-centered magnetic field.

6. SIMPLIFIED EVALUATION OF THE LORENTZ FORCE

If a scheme satisfies conditions (11), then the divergence discretization of the x component of the Lorentz force can be simplified to

$$\begin{aligned}
&\frac{F^x_{i+1/2,j} - F^x_{i-1/2.j}}{\Delta x} + \frac{F^y_{i,j+1/2} - F^y_{i,j-1/2}}{\Delta y} \\
&= \frac{B^y \delta_x B^y - B^x \delta_x B^x}{\Delta x} - \frac{B^x \delta_y B^y + B^y \delta_y B^x}{\Delta y} \\
&= \left(\frac{\delta_x B^y}{\Delta x} - \frac{\delta_y B^x}{\Delta y}\right) B^y = J^z B^y = -(\mathbf{J} \times \mathbf{B})_x \qquad (16)
\end{aligned}$$

and similarly the y component can be simplified to $-(\mathbf{J} \times \mathbf{B})_y$, where the current density has to be defined as

$$\begin{aligned}
J^z &= \frac{\delta_x B^y}{\Delta x} - \frac{\delta_y B^x}{\Delta y} \\
&= + \frac{B^y_{i+1/2,j+1/2} + B^y_{i+1/2,j-1/2} - B^y_{i-1/2,j+1/2} - B^y_{i-1/2,j-1/2}}{2\Delta x} \\
&\quad - \frac{B^x_{i+1/2,j+1/2} + B^x_{i-1/2,j+1/2} - B^x_{i+1/2,j-1/2} - B^x_{i-1/2,j-1/2}}{2\Delta y} \qquad (17)
\end{aligned}$$

in terms of the primary-corner centered magnetic field variables. Of course **B** in the cross-product $\mathbf{J} \times \mathbf{B}$ must be the average of the corner values according to (11b), (11c), and (13).

Note that if the conditions (11) are not satisfied, or if **J** and **B** are discretized in a different way, the simple $\mathbf{J} \times \mathbf{B}$ form of the Lorentz force will be orthogonal to **B**, but it will not be a conservative discretization for the momentum.

7. CONCLUSIONS

In this note I have shown how an orthogonal and conservative discretization of the Lorentz force can be constructed. Although the discussion was restricted to 2D Cartesian grids, it can be trivially generalized to 3D grids. I have not attempted a generalization to non-Cartesian or unstructured grids. It is conceivable that the MUCT schemes [7], which are a class of conservative constrained transport-type schemes on unstructured grids, can be modified to maintain the orthogonality property too. In [11] we discuss how a vertex-based discretization for $\mathbf{B}$ can be used in adaptive grids, but the discrete orthogonality condition may not be easy to satisfy at resolution changes.

It is also unlikely that this discretization can be combined with Strang-type directional splitting [9], because that would require that the F and G fluxes in (6) be evaluated differently. For example, the F flux would be based on $\mathbf{B}^n$ at time level n, while the G flux would use magnetic field $\mathbf{B}^{n+1/2}$ with the contribution of the x directional fluxes already included. Since the x fluxes for the magnetic field depend on the electric field $\mathbf{E} = -\mathbf{u} \times \mathbf{B}$, the velocity would appear as new independent variables in the orthogonality condition (5). It is rather unlikely that the contribution of velocity would be easy to cancel out.

A definite drawback of the discretizations discussed in this paper is the lack of upwinding. All the difference formulas are of the central difference type. The only freedom is in the definition of the electric field in the discrete induction equation (15). Therefore it is not unlikely that an implementation of this type of scheme would exhibit oscillations. The evaluation or comparison of this discretization with alternatives, however, is beyond the scope of this note.

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